Free monotone transport without a trace

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October 30, 2013

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- For an N-tuple $X = (X_1, \ldots, X_N)$, φ_X: $\mathbb{C}\langle t_1, \ldots, t_N \rangle \ni p(t_1, \ldots, t_N) \mapsto \varphi(p(X_1, \ldots, X_N))$.

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- All random variables in this talk will be self-adjoint and non-commutative.

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- *Transport from* φ_X *to* ψ_Z is $Y = (Y_1, \ldots, Y_N) \subset W^*(X_1, \ldots, X_N)$ so that

$$\varphi(p(Y_1, \ldots, Y_N)) = \psi(p(Z_1, \ldots, Z_N)) \qquad \forall p \in \mathbb{C}\langle t_1, \ldots, t_N\rangle;$$

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  that is, $\psi_Z = \varphi_Y$.
- Implies $(W^*(Y_1, \ldots, Y_N), \varphi) \cong (W^*(Z_1, \ldots, Z_N), \psi)$.
- And there is a state-preserving embedding of $W^*(Z_1, \ldots, Z_N)$ into $W^*(X_1, \ldots, X_N)$.

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- Can assume $A = \text{diag}\{A_1, \ldots, A_L, 1 \ldots, 1\}$ with

$$A_k = \frac{1}{2} \begin{pmatrix} \lambda_k + \lambda_k^{-1} & -i(\lambda_k - \lambda_k^{-1}) \\ i(\lambda_k - \lambda_k^{-1}) & \lambda_k + \lambda_k^{-1} \end{pmatrix} \qquad \forall k = 1, \ldots, L$$

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- Then $\text{spectrum}(A) = \{1, \lambda_1^{\pm 1}, \ldots, \lambda_L^{\pm 1}\}$, $A^T = A^{-1}$, $(A^{it})^* = (A^{it})^T = A^{-it}$, and

$$\sum_{j=1}^{N} |[A]_{ij}| \leq \max\{1, \lambda_1^{\pm 1}, \ldots, \lambda_L^{\pm 1}\} \leq \|A\| \qquad \forall i = 1, \ldots, N.$$

- Let $\mathcal{H}_\mathbb{C} = \mathcal{H}_\mathbb{R} + i\mathcal{H}_\mathbb{R}$ and define

$$\langle x, y\rangle_U = \left\langle \frac{2}{1+A^{-1}}x, y \right\rangle, \qquad x, y \in \mathcal{H}_\mathbb{C}.$$

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- The *q-Fock space* $\mathcal{F}_q(\mathcal{H})$ is the completion of $\mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}^{\otimes n}$ with respect to the inner product

$$\langle f_1 \otimes \cdots \otimes f_n, g_1 \otimes \cdots \otimes g_m \rangle_{U,q} = \delta_{n=m} \sum_{\pi \in S_n} q^{i(\pi)} \left\langle f_1, g_{\pi(1)} \right\rangle_U \cdots \left\langle f_n, g_{\pi(n)} \right\rangle_U$$

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- In particular, $\mathcal{F}_0(\mathcal{H})$ is the usual Fock space $\mathcal{F}(\mathcal{H})$.

- For $f \in \mathcal{H}$ we densely define the *left q-creation operator* $l_q(f) \in \mathcal{B}(\mathcal{F}_q(\mathcal{H}))$ by

$$l_q(f)\Omega = f$$
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- We let $s_q(f) = l_q(f) + l_q(f)^*$, and $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)'' = W^*(s_q(f) \colon f \in \mathcal{H}_\mathbb{R})$.

- Ω is cyclic and separating for $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$ and hence the vector state $\varphi(\cdot) = \langle \Omega, \cdot\, \Omega \rangle_{U,q}$ is a faithful, non-degenerate state (*free quasi-free state*

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- With respect to the vacuum vector state φ, the X_j are centered semicircular random variables of variance 1, but aren't free unless $U_t = id$.
- Application of result: for small values of $|q|$, $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$ is isomorphic to M.

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- Using the vector notation $X = (X_1, \ldots, X_N)$ we have $\sigma_z^\varphi(X) = A^{iz} X$.
- KMS condition:

$$\varphi(X_j P) = \varphi(P\sigma_{-i}(X_j)) = \varphi(P[AX]_j)$$
$$\varphi(PX_j) = \varphi(\sigma_i(X_j)P) = \varphi([A^{-1}X]_j P).$$

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$$\|P\|_R := \sum_{n=0}^{\deg(P)} \sum_{|\underline{j}|=n} |c(\underline{j})| R^n = \sum_n \|\pi_n(P)\|_R,$$

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- If $R \geq 2 \geq \|X_j\|$, then $\mathscr{P}^{(R)} \subset M$.

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- On $\left(\mathscr{P}^{(R)}\right)^N$ and $\left(\mathscr{P}^{(R,\sigma)}\right)^N$ we use the max-norm, which we still denote $\|\cdot\|_R$ and $\|\cdot\|_{R,\sigma}$.

- Let $\delta_j \colon \mathscr{P} \to \mathscr{P} \otimes \mathscr{P}^{op}$ be Voiculescu's free difference quotients, defined by $\delta_j(X_k) = \delta_{j=k} 1 \otimes 1$ and the Leibniz rule.

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- As a $\mathscr{P} - \mathscr{P}$ bimodule: $c \cdot (a \otimes b) = (ca) \otimes b$ and $(a \otimes b) \cdot c = a \otimes (bc)$

- For $j, k \in \{1, \ldots, N\}$ denote

$$\alpha_{jk} = \left[\frac{2}{1+A}\right]_{jk} = \varphi(X_k X_j),$$

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- The modular group interacts with ∂_j as follows:

$$(\sigma_i \otimes \sigma_i) \circ \partial_j \circ \sigma_{-i} = \bar{\partial}_j$$

- For $P = (P_1, \ldots, P_N) \in \mathscr{P}^N$ define $\mathscr{J} P, \mathscr{J}_\sigma P \in M_N(\mathscr{P} \otimes \mathscr{P}^{op})$ by

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- Examples:

$$[\mathscr{J} X]_{jk} = \delta_k X_j = \delta_{k=j} 1 \otimes 1 = [1]_{jk}$$
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- A simple computation reveals $\mathscr{J} P = \mathscr{J}_\sigma P \# \mathscr{J}_\sigma X^{-1}$ for all $P \in (\mathscr{P}^{(R)})^N$.

- For each j we define the *j-th σ-cyclic derivative* $\mathscr{D}_j : \mathscr{P} \to \mathscr{P}$ by

$$\mathscr{D}_j(X_{k_1} \cdots X_{k_n}) = \sum_{l=1}^{n} \alpha_{jk_l} \sigma_{-i}(X_{k_{l+1}} \cdots X_{k_n}) X_{k_1} \cdots X_{k_{l-1}},$$

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$$V_0 = \frac{1}{2}\sum_{j,k=1}^{N}\left[\frac{1+A}{2}\right]_{jk} X_k X_j \in \mathscr{P}_{c.s.}^{(R,\sigma)}$$

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  then $\mathscr{D}V_0 = (X_1,\ldots,X_N) = X$.
- Can also define $\bar{\mathscr{D}}_j$ so that $(\mathscr{D}_j P)^* = \bar{\mathscr{D}}_j(P^*)$.

- Given $V \in \mathscr{P}_{c.s.}^{(R,\sigma)}$, we say that a state ψ on $W^*(X_1, \ldots, X_N)$ satisfies the *Schwinger-Dyson equation with potential V* if

$$\psi(\mathscr{D}V\#P) = \psi \otimes \psi^{op} \otimes \mathrm{Tr}(\mathscr{J}_\sigma P) \qquad \forall P \in \mathscr{P}^{(R)},$$

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  in which case we call ψ the *free Gibbs state with potential V*, and may denote it φ_V.
- The state φ_V is unique provided $\|V - V_0\|_{R,\sigma}$ is small enough.
- The vacuum vector state $\varphi = \varphi_{V_0}$.
- Consequently, $X = \mathscr{J}_\sigma^*(1)$, where $1 \in M_N(\mathscr{P} \otimes \mathscr{P}^{op})$ is the identity matrix.

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- By exploiting the Schwinger-Dyson equation, we will construct $Y = (Y_1, \ldots, Y_N) \subset (M, \varphi)$ of the form $Y_j = X_j + f_j$ whose law induced by φ is also the free Gibbs state with potential V.

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- Provided $\|W\|_{R,\sigma}$ is small enough, the free Gibbs state with potential $V_0 + W$ will be unique and therefore we will have transport from φ_X to ψ_Z.

- Suppose $Y = (Y_1, \dots, Y_N)$ with $Y_j = X_j + f_j$ and $f_j \in \mathscr{P}^{(R)}$, assume assume that φ_Y satisfies the Schwinger-Dyson equation with potential $V = V_0 + W$. Then

$$\begin{aligned}(\mathscr{J}_\sigma)^*_Y(1) &= \mathscr{D}_Y(V_0(Y) + W(Y)) \\ &= Y + (\mathscr{D}W)(Y) \\ &= X + f + (\mathscr{D}W)(X + f)\end{aligned} \tag{1}$$

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- Need to write the left-hand side in terms of X.

- Using a change of variables argument, the Schwinger-Dyson equation (1) is equivalent to

$$\mathscr{J}_\sigma^* \circ (1 \otimes \sigma_i) \left(\frac{1}{1+B} \right) = X + f + (\mathscr{D}W)(X+f), \tag{2}$$

where $B = \mathscr{J}_\sigma f \# \mathscr{J}_\sigma X^{-1}$.

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where $B = \mathscr{J}_\sigma f \# \mathscr{J}_\sigma X^{-1}$.

- Using identities $\frac{1}{1+x} = 1 - \frac{x}{1+x}$ and $\frac{x}{1+x} = x - \frac{x^2}{1+x}$ and multiplying by $1+B$, (2) becomes

$$\begin{aligned} &- \mathscr{J}_\sigma^* \circ (1 \otimes \sigma_i)(B) - f \\ &\qquad = \mathscr{D}(W(X+f)) + B \# f + B \# \mathscr{J}_\sigma^* \circ (1 \otimes \sigma_i) \left(\frac{B}{1+B} \right) \qquad (3) \\ &\qquad\qquad - \mathscr{J}_\sigma^* \circ (1 \otimes \sigma_i) \left(\frac{B^2}{1+B} \right), \end{aligned}$$

Lemma 2.1

Let $g = g^* \in \mathscr{P}_\varphi^{(R,\sigma)}$ *and let* $f = \mathscr{D}g$. *Then for any* $m \geq -1$ *we have:*

$$B\#\mathscr{J}_\sigma^* \circ (1 \otimes \sigma_i)(B^{m+1}) - \mathscr{J}_\sigma^* \circ (1 \otimes \sigma_i)(B^{m+2}) \tag{4}$$
$$= \frac{1}{m+2}\mathscr{D}\left[(\varphi \otimes 1) \circ Tr_{A^{-1}} + (1 \otimes \varphi) \circ Tr_A\right](B^{m+2})$$

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Proof.

We prove the equivalence weakly by taking inner products against $P \in (\mathscr{P}^{(R)})^N$. Denote the left-hand side by E_L and the right-hand side by E_R.

Proof of Lemma 2.1 (conti.)

$$\left\langle P,\ B\#\mathscr{J}_\sigma^* \circ (1\otimes\sigma_i)(B^{m+1})\right\rangle_\varphi$$

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$$\langle P,\ B\#\mathscr{J}_\sigma^* \circ (1\otimes\sigma_i)(B^{m+1})\rangle_\varphi$$
$$= \sum_{j,k=1}^{N} \varphi\left(P_j^* \cdot B_{jk}\#\left[\mathscr{J}_\sigma^* \circ (1\otimes\sigma_i)(B^{m+1})\right]_k\right)$$

Proof of Lemma 2.1 (conti.)

$$\begin{aligned}
&\left\langle P,\ B\#\mathscr{J}_\sigma^* \circ (1\otimes\sigma_i)(B^{m+1})\right\rangle_\varphi \\
&\quad = \sum_{j,k=1}^{N} \varphi\left(P_j^* \cdot B_{jk}\#\left[\mathscr{J}_\sigma^* \circ (1\otimes\sigma_i)(B^{m+1})\right]_k\right) \\
&\quad = \sum_{j,k=1}^{N} \varphi\left((\sigma_i\otimes 1)(B_{jk}^\diamond)\#P_j^* \cdot \left[\mathscr{J}_\sigma^* \circ (1\otimes\sigma_i)(B^{m+1})\right]_k\right)
\end{aligned}$$

Proof of Lemma 2.1 (conti.)

$$
\begin{aligned}
&\left\langle P,\ B\#\mathscr{J}_\sigma^*\circ(1\otimes\sigma_i)(B^{m+1})\right\rangle_\varphi \\
&\quad=\sum_{j,k=1}^{N}\varphi\left(P_j^*\cdot B_{jk}\#\left[\mathscr{J}_\sigma^*\circ(1\otimes\sigma_i)(B^{m+1})\right]_k\right) \\
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&\quad=\left\langle(1\otimes\sigma_{-i})(B^*)\#P,\mathscr{J}_\sigma^*\circ(1\otimes\sigma_i)(B^{m+1})\right\rangle_\varphi
\end{aligned}
$$

Proof of Lemma 2.1 (conti.)

$$
\begin{aligned}
&\left\langle P,\ B\#\mathscr{J}_\sigma^* \circ (1\otimes\sigma_i)(B^{m+1})\right\rangle_\varphi \\
&= \sum_{j,k=1}^{N} \varphi\left(P_j^* \cdot B_{jk}\#\left[\mathscr{J}_\sigma^* \circ (1\otimes\sigma_i)(B^{m+1})\right]_k\right) \\
&= \sum_{j,k=1}^{N} \varphi\left((\sigma_i\otimes 1)(B_{jk}^\diamond)\#P_j^* \cdot \left[\mathscr{J}_\sigma^* \circ (1\otimes\sigma_i)(B^{m+1})\right]_k\right) \\
&= \left\langle (1\otimes\sigma_{-i})(B^*)\#P, \mathscr{J}_\sigma^* \circ (1\otimes\sigma_i)(B^{m+1})\right\rangle_\varphi \\
&= \left\langle [\mathscr{J}_\sigma X^{-1}\#\hat{\sigma}_i(\mathscr{J}_\sigma f)]\#P, \mathscr{J}_\sigma^* \circ (1\otimes\sigma_i)(B^{m+1})\right\rangle_\varphi
\end{aligned}
$$

where $\hat{\sigma}_i = \sigma_i \otimes \sigma_{-i}$.

Proof of Lemma 2.1 (conti.)

Hence if $\phi = \varphi \otimes \varphi^{op} \otimes \mathrm{Tr}$ then

$$\langle P, E_L \rangle_\varphi = \left\langle \mathscr{J}_\sigma X^{-1} \# \mathscr{J}_\sigma \left\{ \hat{\sigma}_i(\mathscr{J}_\sigma f) \# P \right\}, (1 \otimes \sigma_i)(B^{m+1}) \right\rangle_\phi - \left\langle \mathscr{J}_\sigma P, (1 \otimes \sigma_i)(B^{m+2}) \right\rangle_\phi .$$

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The "product rule" simplifies the right-hand side to simplify to

$$\langle P, E_L \rangle_\varphi = \left\langle Q^P, \mathscr{J}_\sigma X^{-1} \# (1 \otimes \sigma_i)(B^{m+1}) \right\rangle_\phi ,$$

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where, if $a \otimes b \otimes c \#_1 \xi = (a \xi b) \otimes c$ and $a \otimes b \otimes c \#_2 \xi = a \otimes (b \xi c)$, then

$$[Q^P]_{jk} = \sum_{l=1}^{N} (\partial_k \otimes 1) \circ \hat{\sigma}_i \circ \partial_l (f_j) \#_2 P_l + (1 \otimes \partial_k) \circ \hat{\sigma}_i \circ \partial_l (f_j) \#_1 P_l$$

Proof of Lemma 2.1 (conti.)

So we have

$$\langle E_L, P\rangle_\varphi = \phi(Q^P \# \mathscr{J}_\sigma X^{-1} \# B^{m+1})$$

Proof of Lemma 2.1 (conti.)

Next for $u = 1, \ldots, m+2$ let R_u be the matrix will all zero entries except $[R_u]_{i_u j_u} = a_u \otimes b_u$.

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$$\sum_k \varphi(\bar{\mathscr{D}}_k(\varphi \otimes 1)\mathrm{Tr}_{A^{-1}}(R_1 \cdots R_{m+2}) P_k)$$

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$$\sum_k \varphi(\bar{\mathscr{D}}_k(\varphi \otimes 1)\mathrm{Tr}_{A^{-1}}(R_1 \cdots R_{m+2})P_k)$$

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$$= \sum_{k,u} C\varphi(\sigma_i(a_u \cdots a_{m+2})a_1 \cdots a_{u-1})$$

$$\times \varphi(b_{u-1} \cdots b_1 \sigma_i(b_{m+2} \cdots b_{u+1}) \cdot \hat{\sigma}_i \circ \partial_k(b_u)\#P_k)$$

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&= \sum_{k,u} C\varphi(\sigma_i(a_u \cdots a_{m+2})a_1 \cdots a_{u-1}) \\
&\quad \times \varphi(b_{u-1} \cdots b_1 \sigma_i(b_{m+2} \cdots b_{u+1}) \cdot \hat{\sigma}_i \circ \partial_k(b_u)\# P_k) \\
&= \sum_u \phi(\Delta_{(1,P)}(R_u)(\sigma_i \otimes \sigma_i)(R_{u+1} \cdots R_{m+2})A^{-1} R_1 \cdots R_{u-1})
\end{aligned}$$

Proof of Lemma 2.1 (conti.)

Where for an arbitrary matrix O

$$[\Delta_{(1,P)}(O)]_{jk} = \sum_{l} \sigma_i \otimes (\hat{\sigma}_i \circ \partial_l)([O]_{jk}) \#_2 P_l.$$

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$$= \sum_u \phi(\Delta_{(1,P)}(B)(\sigma_i \otimes \sigma_i)(B^{m+2-u})A^{-1}B^{u-1})$$

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Proof of Lemma 2.1 (conti.)

Where for an arbitrary matrix O

$$[\Delta_{(1,P)}(O)]_{jk} = \sum_l \sigma_i \otimes (\hat{\sigma}_i \circ \partial_l)([O]_{jk}) \#_2 P_l.$$

Replacing R_u with B for each u and using $(\sigma_i \otimes \sigma_i)(B)A^{-1} = A^{-1}B$ turns the previous equation into

$$\begin{aligned}\langle \mathscr{D}(\varphi \otimes 1)\ \mathrm{Tr}_{A^{-1}}(B^{m+2}), P\rangle_\varphi \\ &= \sum_u \phi(\Delta_{(1,P)}(B)(\sigma_i \otimes \sigma_i)(B^{m+2-u})A^{-1}B^{u-1}) \\ &= (m+2)\phi(\Delta_{(1,P)}(B)A^{-1}B^{m+1})\end{aligned}$$

Proof of Lemma 2.1 (conti.)

Similarly,

$$\left\langle \mathscr{D}(1 \otimes \varphi)\mathrm{Tr}_A(B^{m+2}), P \right\rangle_\varphi = (m+2)\phi(\Delta_{(2,P)}(B)AB^{m+1}),$$

where

$$[\Delta_{(2,P)}(O)]_{jk} = \sum_l (\hat{\sigma}_i \circ \partial_l) \otimes \sigma_{-i}([O]_{jk}) \#_1 P_l.$$

Proof of Lemma 2.1 (conti.)

Similarly,

$$\left\langle \mathscr{D}(1 \otimes \varphi)\mathrm{Tr}_A(B^{m+2}), P \right\rangle_\varphi = (m+2)\phi(\Delta_{(2,P)}(B)AB^{m+1}),$$

where

$$[\Delta_{(2,P)}(O)]_{jk} = \sum_l (\hat{\sigma}_i \circ \partial_l) \otimes \sigma_{-i}([O]_{jk}) \#_1 P_l.$$

To finish the proof we simply verify that

$$Q^P \# \mathscr{J}_\sigma X^{-1} = \Delta_{(1,P)}(B)A^{-1} + \Delta_{(2,P)}(B)A,$$

which follows from their definitions after decomposing the various derivations as linear combinations of the free difference quotients δ_k. □

Define

$$\mathscr{N}(X_{\underline{i}}) = |\underline{i}| X_{\underline{i}}$$
$$\Sigma(X_{\underline{i}}) = \frac{1}{|\underline{i}|} X_{\underline{i}}$$

- Recall $f = \mathscr{D}g$, and $B = \mathscr{J}_\sigma f \# \mathscr{J}_\sigma X^{-1} = \mathscr{J} f$. Set

$$Q(g) = [(1 \otimes \varphi) \circ \mathrm{Tr}_A + (\varphi \otimes 1) \circ \mathrm{Tr}_{A^{-1}}](B - \log(1 + B)),$$

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- Then by comparing power series the previous lemma implies

$$\mathscr{D}Q(g) = B \# \mathscr{J}_\sigma^* \circ (1 \otimes \sigma) \left(\frac{B}{1+B} \right) - \mathscr{J}_\sigma^* \circ (1 \otimes \sigma_i) \left(\frac{B^2}{1+B} \right).$$

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Lemma 2.2

Assume $f = \mathscr{D}g$ for $g = g^* \in \mathscr{P}_\varphi^{(R,\sigma)}$ and $\| \mathscr{J} \mathscr{D}g \|_{R \otimes_\pi R} < 1$. Then equation (3) is equivalent to

$$\begin{aligned} \mathscr{D}\{[(\varphi \otimes 1) \circ Tr_{A^{-1}} + (1 \otimes \varphi) \circ Tr_A](\mathscr{J}\mathscr{D}g) - \mathscr{N} g\} \\ = \mathscr{D}(W(X + \mathscr{D}g)) + \mathscr{D}Q(g) + (\mathscr{J}\mathscr{D}g) \# \mathscr{D}g \end{aligned} \tag{5}$$

Corollary 2.3

Let $g \in \mathscr{P}_{c.s.}^{(R,\sigma)}$ and assume that $\|g\|_{R,\sigma} < R^2/2$. Let $S \geq R + \|g\|_{R,\sigma}$. Let $S \geq R + \|g\|_{R,\sigma}$ and let $W \in \mathscr{P}_{c.s.}^{(S)}$. Assume $|\varphi(X_{\underline{j}})| \leq C_0^{|\underline{j}|}$ for all $\underline{j}$ and some $C_0 > 0$ and furthermore that $C_0/R < 1/2$. Let

$$F(g) = -W(X + \mathscr{D}\Sigma g) - \frac{1}{2}\{\mathscr{J}_\sigma X^{-1} \# \mathscr{D}\Sigma g\} \# \mathscr{D}\Sigma g$$
$$+ [(1 \otimes \varphi) \circ Tr_A + (\varphi \otimes 1) \circ Tr_{A^{-1}}](\mathscr{J}\mathscr{D}\Sigma g) - Q(\Sigma g)$$

Then $F(g)$ is a well-defined function from $\mathscr{P}_{c.s.}^{(R,\sigma)}$ to $\mathscr{P}_{\varphi}^{(R,\sigma)}$. In particular, if we fix $0 < \rho \leq 1$ and $R > 4\sqrt{\|A\|}$, then $\|W\|_{R,\sigma} < \frac{\rho}{2N}$ and $\sum_j \|\delta_j(W)\|_{(R+\rho)\otimes_\pi(R+\rho)} < \frac{1}{8}$ imply that

$$E_1 := \left\{ g \in \mathscr{P}_{c.s.}^{(R,\sigma)} : \|g\|_{R,\sigma} < \frac{\rho}{N} \right\} \overset{F}{\mapsto} \left\{ g \in \mathscr{P}_{\varphi}^{(R,\sigma)} : \|g\|_{R,\sigma} < \frac{\rho}{N} \right\} =: E_2$$

and is uniformly contractive with constant $\lambda \leq \frac{1}{2}$ on E_1.

Define

$$\mathscr{S}(X_{\underline{j}}) = \frac{1}{|\underline{j}|} \sum_{n=0}^{|\underline{j}|-1} \rho^n(X_{\underline{j}}),$$

and $\mathscr{S}(c) = c$ for $c \in \mathbb{C}$.

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Denote

$$\Pi = id - \pi_0$$

Proposition 2.4

Assume that for some $R > 4\sqrt{\|A\|}$ *and some* $0 < \rho \leq 1$, $W \in \mathscr{P}_{c.s.}^{(R+\rho,\sigma)} \subset \mathscr{P}_{c.s.}^{(R,\sigma)}$ *and that* $\|W\|_{R,\sigma} < \frac{\rho}{2N}$ *and* $\sum_j \|\delta_j(W)\|_{(R+\rho)\otimes_\pi(R+\rho)} < \frac{1}{8}$. *Then there exists* $\hat{g}$ *and* $g = \Sigma\hat{g}$ *such that:*

(i) $\hat{g}, g \in \mathscr{P}_{c.s.}^{(R,\sigma)}$

(ii) $\hat{g}$ *satisfies* $\hat{g} = \mathscr{S}\Pi F(\hat{g})$ *and* g *satisfies*

$$\mathscr{N} g = \mathscr{S}\Pi\Big[-W(X + \mathscr{D}g) - \frac{1}{2}\{\mathscr{J}_\sigma X^{-1}\#\mathscr{D}g\}\#\mathscr{D}g - Q(g) + [(1\otimes\varphi)\circ Tr_A + (\varphi\otimes 1)\circ Tr_{A^{-1}}](\mathscr{J}\mathscr{D}g)\Big]$$

(iii) *If* W *is self-adjoint, then so are* $\hat{g}$ *and* g.

Proof.

Set $\hat{g}_0 = W(X_1, \ldots, X_N) \in E_1$ and for each $k \in \mathbb{N}$, $\hat{g}_k := \mathscr{S}\Pi F(\hat{g}_{k-1})$.

Proof.

Set $\hat{g}_0 = W(X_1, \ldots, X_N) \in E_1$ and for each $k \in \mathbb{N}$, $\hat{g}_k := \mathscr{S}\Pi F(\hat{g}_{k-1})$. We have

$$E_1 \xrightarrow{F} E_2 \xrightarrow{\mathscr{S}\Pi} E_1,$$

Proof.

Set $\hat{g}_0 = W(X_1, \dots, X_N) \in E_1$ and for each $k \in \mathbb{N}$, $\hat{g}_k := \mathscr{S}\Pi F(\hat{g}_{k-1})$. We have

$$E_1 \xrightarrow{F} E_2 \xrightarrow{\mathscr{S}\Pi} E_1,$$

so that $\{\hat{g}_k\}_{k\in\mathbb{N}}$ is a sequence in E_1 with $\|\hat{g}_k - \hat{g}_{k-1}\|_{R,\sigma} \leq \frac{1}{2}\|\hat{g}_{k-1} - \hat{g}_{k-2}\|_{R,\sigma}$.

Proof.

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We note $\hat{g} \neq 0$ since $\mathscr{S}\Pi F(0) = \mathscr{S}\Pi(W) = W \neq 0$.

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$\|\hat{g}_k - \hat{g}_{k-1}\|_{R,\sigma} \leq \frac{1}{2}\|\hat{g}_{k-1} - \hat{g}_{k-2}\|_{R,\sigma}$. Thus $\{\hat{g}_k\}$ converges to some $\hat{g} \in \mathscr{P}^{(R,\sigma)}_{c.s.}$ which is a fixed point of $\mathscr{S}\Pi F$.
We note $\hat{g} \neq 0$ since $\mathscr{S}\Pi F(0) = \mathscr{S}\Pi(W) = W \neq 0$.
Setting $g = \Sigma\hat{g}$ (so $\mathscr{N} g = \hat{g}$), yields (i) and (ii).
If W is self adjoint then it follows that $\mathscr{S}\Pi F(h)^* = \mathscr{S}\Pi F(h)$ for $h = h^*$ and hence the sequence $\{\hat{g}_k\}$ is self-adjoint. □

Theorem 2.5

Let $R' > R > 4\sqrt{\|A\|}$. Then there exists a constant $C > 0$ depending only on R, R', and N so that whenever $W = W^* \in \mathscr{P}^{(R',\sigma)}_{c.s.}$ satisfies $\|W\|_{R'+1,\sigma} < C$, there exists $f \in \mathscr{P}^{(R)}$ which satisfies equation (2). In addition, $f = \mathscr{D}g$ for $g \in \mathscr{P}^{(R,\sigma)}_{c.s.}$. The solution $f = f_W$ satisfies $\|f_W\|_R \to 0$ as $\|W\|_{R'+1,\sigma} \to 0$.

Theorem 2.6

Let φ be a free quasi-free state corresponding to A, and let $X_1, \ldots, X_N \in (M, \varphi)$ be self-adjiont elements whose law φ_X is the unique Gibbs law with potential V_0. Let $R' > R > 4\sqrt{\|A\|}$. Then there exists $C > 0$ depending only on R, R', and N so that whenever $W = W^* \in \mathscr{P}^{(R'+1,\sigma)}_{c.s.}$ satisfies $\|W\|_{R'+1,\sigma} < C$, there exists $G \in \mathscr{P}^{(R,\sigma)}_{c.s.}$ so that:

1. *If we set $Y_j = \mathscr{D}_j G$ then $Y_1, \ldots, Y_N \in \mathscr{P}^{(R)}$ has the law φ_V, with $V = V_0 + W$;*
2. *$X_j = H_j(Y_1, \ldots, Y_N)$ for some $H_j \in \mathscr{P}^{(R)}$;*
3. *if $R' > R\sqrt{\|A\|}$ then $(\sigma_{i/2} \otimes 1)(\mathscr{J}_\sigma \mathscr{D} G) \geq 0$.*

In particular, there are state-preserving isomorphisms

$$C^*(\varphi_V) \cong \Gamma(\mathcal{H}_\mathbb{R}, U_t), \qquad W^*(\varphi_V) \cong \Gamma(\mathcal{H}_\mathbb{R}, U_t)''.$$

- Let $M_q = \Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$, so that M_q is generated by $Z_j = s_q(e_j)$.

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- Can identify $L^2(M_q \bar{\otimes} M_q^{op})$ with $HS(\mathcal{F}_q(\mathcal{H}))$ via $a \otimes b^{op} \mapsto \langle b\Omega, \cdot\,\Omega\rangle\, a\Omega$. For example $1 \otimes 1^\circ \mapsto P_0$.

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- ξ_j are called the conjugate variables of $Z_1, \ldots, Z_N$ with respect to $\partial_1, \ldots, \partial_N$ and in fact are merely $\partial_j^*(1 \otimes 1)$.
- Do not necessarily exist, but for small enough $|q|$ they do with $\xi_j = \partial_j^{(q)*} \circ \hat{\sigma}_{-i}(\left[\Xi_q^{-1}\right]^*)$.

- Define

$$V = \Sigma \left(\sum_{j,k=1}^{N} \left[\frac{1+A}{2} \right]_{jk} \xi_k Z_j \right) \qquad V_0 = \frac{1}{2} \sum_{j,k=1}^{N} \left[\frac{1+A}{2} \right]_{jk} Z_k Z_j,$$

and let $W = V - V_0$.

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  and let $W = V - V_0$.

- Then $\mathscr{D}_{Z_j} V = \xi_j$ and so the vacuum state φ satisfies the Schwinger-Dyson equation with potential V:

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- Turns out it suffices to show $\|(\sigma_i \otimes 1)(\Xi_q^{-1}) - 1 \otimes 1\|_{R\otimes_\pi R}$ can be made small.
- By adapting the estimates of Dabrowski in [1], can show this quantity tends to zero as $|q| \to 0$.

Theorem 3.1

For $\mathcal{H}_\mathbb{R}$ finite dimensional, then there exists $\epsilon > 0$ depending on N such that $|q| < \epsilon$ implies

$$\Gamma_q(\mathcal{H}_\mathbb{R}, U_t) \cong \Gamma_0(\mathcal{H}_\mathbb{R}, U_t) \qquad \text{and} \qquad \Gamma_q(\mathcal{H}_\mathbb{R}, U_t)'' \cong \Gamma_0(\mathcal{H}_\mathbb{R}, U_t)''.$$

In particular, if G is the multiplicative subgroup of $\mathbb{R}_+^\times$ generated by the spectrum of A then

$$\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)'' \text{ is a factor of type } \begin{cases} III_1 & \text{if } G = \mathbb{R}_+^\times \\ III_\lambda & \text{if } G = \lambda^\mathbb{Z},\ 0 < \lambda < 1 \\ II_1 & \text{if } G = \{1\}. \end{cases}$$

Moreover $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$ is full.

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